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On the K Functional between L^1 and L^2 and Some Other K Functionals

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DEDICATED TO THE MEMORY OF GÉZA FREUD

PREFACE BY JAAK PEETRE

Géza Freud was one of my oldest mathematical acquaintances. One day in the spring of '57, which was my first year as a graduate student, I was sitting in the reading room, and Géza, then a visitor in Lund, approached me and started a conversation with these German words: "Haben Sie publiziert?" I am proud that Géza, much later, became interested in the K functional.

1. THE CASES (L^1, L^2) AND (L^1, L^p)

Recall that if A_0 and A_1 are Banach spaces, both continuously imbedded in a Hausdorff topological vector space A, and if $f \in A_0 + A_1$ (hull) and $0 < t < \infty$, one puts

$$K(t, f) = K(t, f; A_0, A_1) = \inf(\| f_0 \|_{A_0} + t \| f_1 \|_{A_1})$$

where the inf extends over all decompositions $f = f_0 + f_1$ of f with $f_0 \in A_0$, $f_1 \in A_1$. For more details about the K (and the J) functional see, e. g., [1].

All spaces considered below are spaces of measurable functions over

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some measure space. We begin with the simplest case $A_0 = L^1$, $A_1 = L^2$. It is then no essential loss of generality to assume that f is a nondecreasing function on $(0, \infty)$ (equipped with the usual measure). This is the result.

THEOREM 1. *Let f be a decreasing positive function on* $(0, \infty)$ *in* $L^1 + L^2$. *Then*

$$K(t, f; L^1, L^2) = \int_0^\lambda f(x)\, dx + \int_\lambda^\infty (f(x))^2/f(\lambda)\, dx$$

where λ is determined by

$$t^2 = \lambda + \int_\lambda^\infty (f(x)/f(\lambda))^2\, dx.$$

Proof. In view of the duality between K and J (see [1]) it suffices to estimate $\int_0^\infty f(x)\, g(x)\, dx$, where g, too, is decreasing in the same interval, under the side conditions $g(x) \leqslant 1$, $\int_0^\infty (g(x))^2\, dx \leqslant t^2$. Define

$$\begin{aligned} g^\#(x) &= Af(\lambda), \qquad x \leqslant \lambda \\ &= Af(x), \qquad x > \lambda. \end{aligned}$$

If $A = 1/f(\lambda)$ clearly the first requirement is met. To fullfil the second one we have to consider an integral:

$$\begin{aligned} \int (g^\#(x))^2\, dx &= A^2\left((f(\lambda))^2 \lambda + \int_\lambda^\infty (f(x))^2\, dx \right) \\ &= \lambda + \int_\lambda^\infty (f(x)/f(\lambda))^2\, dx. \end{aligned}$$

This clearly gives the condition in the theorem.

CLAIM. $0 \leqslant (g^\# - g)(2Af - (g^\# + g))$ pointwise.

To prove the claim we distinguish two cases.

Case 1. $g(x) \leqslant g^\#(x)$. Then $g(x) + g^\#(x) \leqslant 2g^\#(x) \leqslant 2Af(x)$. ∎

Case 2. $g(x) > g^\#(x)$. Now $g^\#(x) = Af(x)$. Hence $g(x) + g^\#(x) > 2g^\#(x) = 2Af(x)$. ∎

Integrate the inequality in the claim!

$$0 \leqslant 2A \int f g^\# - 2A \int f g - \left(\int (g^\#)^2 - \int g^2 \right).$$

In view of the second hypothesis on g the last term here is $\geqslant 0$. Hence

$$2A\int fg \leqslant 2A\int fg^{\#} \qquad \text{or} \qquad \int fg \leqslant \int fg^{\#}.$$

It remains to evaluate the integral

$$\int fg^{\#} = Af(\lambda)\int_0^{\lambda} f(x)\,dx + A\int_{\lambda}^{\infty} (f(x))^2\,dx$$
$$= \int_0^{\lambda} f(x)\,dx + \int_{\lambda}^{\infty} (f(x))^2/f(\lambda)\,dx. \quad \blacksquare$$

It is now an easy matter to extend this result to the more general case $A_0 = L^1$, $A_1 = L^p$, where $1 < p < \infty$.

THEOREM 1′. *Let f be as before but in $L^1 + L^p$, where $1 < p < \infty$. Let $r = p/(p-1)$ be the conjugate exponent. Then*

$$K(t, f; L^1, L^p) = \int_0^{\lambda} f(x)\,dx + \int_{\lambda}^{\infty} (f(x))^p/(f(\lambda))^{p-1}\,dx$$

where λ is determined by

$$t^r = \lambda + \int_{\lambda}^{\infty} (f(x)/f(\lambda))^p\,dx.$$

Proof. We indicate only the essential changes compared to Theorem 1. The side conditions now take the form $g(x) \leqslant 1$, $\int_0^{\infty}(g(x))^r\,dx \leqslant t^r$ and the "best" function $g^{\#}$ is defined by

$$\begin{aligned} g^{\#}(x) &= A(f(\lambda))^{p-1}, && x \leqslant \lambda \\ &= A(f(x))^{p-1}, && x > \lambda. \end{aligned}$$

Finally, the crucial inequality in the claim has to be changed to

$$0 \leqslant (g^{\#} - g)\cdot\left\{rA^{r-1}f - \frac{(g^{\#})^r - g^r}{g^{\#} - g}\right\}.$$

The rest goes through as before. $\blacksquare$

Remarks 1. In the limiting case $p = \infty$ one gets the well-known formula for K functional for the pair (L^1, L^{∞}) (see [1]).

2. Reverting to Theorem 1 and $p = 2$, a special case (essentially $f = x^{-1/2}$ on $(0, 1)$) can be found on pp. 104–105 in [2] under the heading

Valiron–Landau Lemma. In [4] (see especially pp. 630–631), quoted by the author of [2], the result is attributed to Valiron. The reference to the latter is perhaps [5]. The preceding proof is patterned after the one in [2, 4]. It seems, however, that the introduction of the K functional does not shed any new light in this particular function theoretic context, but it is this example which triggered off all this investigation.

3. It is also easy to write down the optimal decomposition $f = f_0 + f_1$ in the K functional for the pair (L^1, L^p). Namely, one simply has to take $f_0(x) = f(x) - f(\lambda)$ for $x \leqslant \lambda$, $= 0$ for $x > \lambda$ so that, consequently, $f_1(x) = f(\lambda)$ for $x \leqslant \lambda$, $= f(x)$ for $x > \lambda$. Indeed, with this choice one finds

$$\begin{aligned} \| f_0 \|_{L^1} + t \| f_1 \|_{L^p} &= \int_0^\lambda (f(x) - f(\lambda))\, dx \\ &\quad + t \left[\int_0^\lambda (f(\lambda))^p \, dx + \int_\lambda^\infty (f(x))^p \, dx \right]^{1/p} \\ &= [\text{sic!}] \int_0^\lambda f(x)\, dx + \int_\lambda^\infty (f(x))^p/(f(\lambda))^{p-1}\, dx. \end{aligned}$$

if λ is as in th. 1′.

2. The Case (L^q, L^p)

Consider now the case $A_0 = L^q$, $A_1 = L^p$, where $1 < q < p < \infty$. We wish to extend the results in Remark 3 to the present situation. Write $s = q/(q-1)$, $r = p/(p-1)$ (conjugate exponents). The problem is thus to maximize the integral $\int fg$ under the side conditions $\int g^s = 1$, $\int g^r = t^r$. A formal application of Langrange's multipliers gives

$$f = mg^{s-1} + ng^{r-1} = f_0 + f_1. \tag{1}$$

Thus we get

$$K = \int fg = m \int g^s + n \int g^r = m + nt^r$$

We have further

$$f_0^q = m^q g^{q(s-1)}, \qquad f_1^p = n^p g^{p(r-1)},$$

$$\int f_0^q = m^q \int g^s = m^q, \qquad \int f_1^p = n^p \int g^r = n^p t^r,$$

$$\| f_0 \|_{L^q} = m, \qquad \| f_1 \|_{L^p} = nt^{r/p} = nt^{r-1}.$$

Thus

$$\|f_0\|_{L^q} + t\|f_1\|_{L^p} = m + tnt^{r-1} = m + nt^r = K = K(t, f; L^q, L^p)$$

and we have the optimal decomposition.

It is also possible to get a more explicit formula. Write $m = ab^{-(s-1)}$, $n = ab^{-(r-1)}$. Then the equation for g becomes

$$(f/a) = (g/b)^{s-1} + (g/b)^{r-1}. \tag{1'}$$

If F denotes the inverse of the map $x \to x^{s-1} + x^{r-1}$, this gives

$$g = bF(f/a). \tag{2}$$

We summarize:

THEOREM 2. *The optimal decomposition $f = f_0 + f_1$ in the functional for the pair (L^q, L^p) is given by*

$$f_0 = a(F(f/a)/b)^{s-1}, \qquad f_1 = a(F(f/a)/b)^{r-1}$$

where a and b are determined so that

$$\int (F(f/a))^s = b^{-s}, \int (F(f/a))^r = t^r b^{-r}. \quad ∎$$

3. Conclusion

If one uses the aproximation $F(x) \approx \min(x^{1/(s-1)}, x^{1/(r-1)})$, one is lead to Holmstedt's formula [3]. In a way, what we do is a step back to the pre-Holmstedt era. Indeed, Holmstedt originally had the thesis assignment to determine the exact K functional for (L^q, L^p) but he had the genius to see that here the approximate formulae are more useful from the practical point of view. But it may be that the present approach will yield something in the case of Orlicz space. The problem is then to maximize the integral $\int fg$ under side conditions of the type $\int \psi_0(g) \leqslant 1$, $\int \psi_1(g/t) \leqslant 1$. A concrete problem: To determine exactly $K(t, f; L^q, L^p)$ if $f = x^{-\lambda}$ on $(0, 1)$ (generalization of Remark 2).

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